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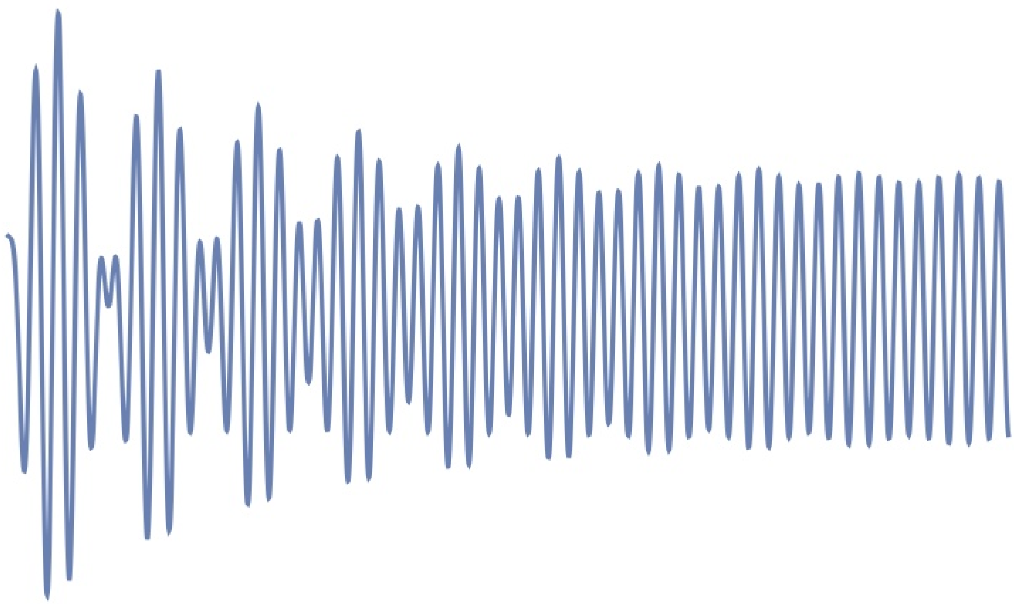

An Elementary Account of Plucked String Clonk – a key part of banjo sound

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The sudden onset of a plucked string's oscillatory force on an acoustic instrument's soundboard excites transients of the soundboard and instrument's body. The transients' frequencies are not directly related to those of the string's vibration. And, on a banjo, they represent a significant fraction of the total sound. An account of the basic physics is provided by a simple extension of the standard solution of the forced, damped harmonic oscillator. (This approach is more elementary and perhaps more physical and transparent than the more professional method using Laplace transforms.)

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I. INTRODUCTION

"Clonk" is the sound of an abrupt, heavy impact. In 1983, Jim Woodhouse[1] used the word specifically to apply to the transient sound on an instrument body excited by the sudden application of a string pluck. As identified by Helmholtz, a violin string is plucked once every cycle of the string fundamental by the stick-slip action of the bow. Woodhouse suggested that the repeated clonk sound might give a significant contribution to the instrument's particular voice. While single clonks on wood-topped instruments are particularly weak and short-lived, on banjos the clonks are significant.[2] Their amplitudes and decay times are substantial fractions of the plucked string sound. However, they are made up of frequencies that are decidedly unrelated to the strings' harmonics.

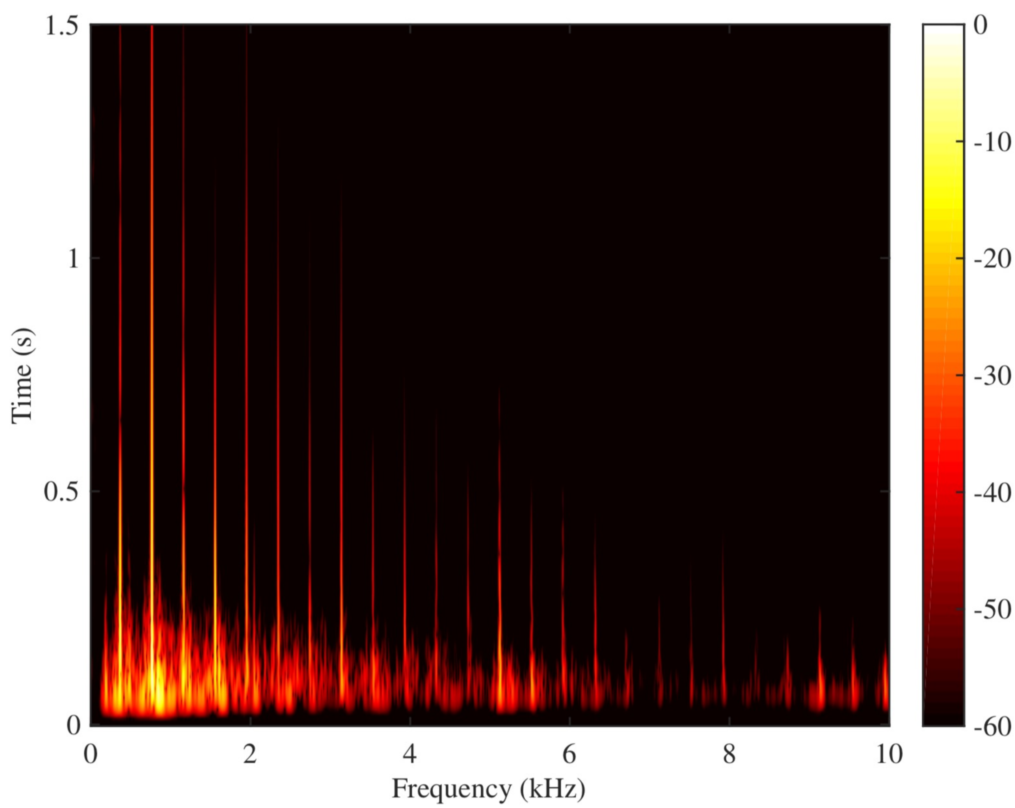

FIG. 1. Spectrogram of a single banjo pluck on the 1st string, 5th fret — from ref. [2]

Fig. 1 shows the spectrogram of a single banjo pluck on the 1st string, 5th fret — taken from ref. [2]. The equally spaced thin lines reflect the string's harmonics driving the head.

In addition, there is a lot of loud stuff at early times. That is the clonk. To appreciate its significance, note that, in normal playing, banjo notes are often not more than about 0.25 seconds apart.

II. THE PLAN

This note explains a way to think about clonks as a simple generalization of the forced, damped harmonic oscillator. A standard physics textbook analysis of that problem is in terms of general and particular solutions and fitting initial conditions. The plucked string is a set of oscillators that drive the set of oscillators representing the soundboard and body. So there is a double sum over drivers and driven. To connect specifically to the clonk, a simplifying approximation is to ignore the decay times of the string vibrations and consider only the decaying soundboard.

III. REMEMBER:

The immediate cause of sound is the vibration of the soundboard and **NOT** the vibration of the strings!

IV. THE ANSWER

You probably know or at least can imagine that the sudden onset of a sinusoidal force driving an oscillator produces a transient with the oscillator's natural (or resonant or free decay) frequency, ω_o. This is in addition to a steady response at the driving frequency, ω. What the analysis offered below demonstrates is that the ratio of the initial amplitudes of the two motions is essentially ω^2/ω_o^2. In general, not much happens at either frequency for $\omega \ll \omega_o$. The strong response is for ω_o comparable to ω. However, this analysis suggests that cases with $\omega \gg \omega_o$ also give a strong excitation of the ω_o transient in spite of a tiny ω persistent response. As always with Fourier asymptotics, this "high" ω behavior persists only to the extent that the onset is infinitely sharp. The actual response dies off rapidly with increasing ω above the scale determined by the time spanned by the onset.

V. THE FORCED, DAMPED HARMONIC OSCILLATOR

The equation of motion for the (ideal) damped harmonic oscillator is linear in its displacement, $x(t)$:

$$F = -kx - \gamma\dot{x} = m\ddot{x}$$

where $\dot{x} = v$, the velocity, and $\ddot{x} = a$, the acceleration. When considering an arbitrary applied force, $F_{\mathrm{external}}(t)$, with "no loss of generality," one can focus on a single sinusoidal applied force $F_\omega(t) = \mathtt{F}_\omega \cos \omega t$. The solution for the general $F_{\mathrm{external}}(t)$ is just the sum of the solutions for the various $F_\omega(t)$'s, where the $F_\omega(t)$'s are the Fourier components of $F_{\mathrm{external}}(t)$.

The solution for a given $F_\omega(t)$ is

$$x_\omega(t) = A_\omega \cos(\omega t + \phi)$$

where $A_\omega/\mathtt{F}_\omega$ and ϕ are *determined* functions of ω, k, γ, and oscillator's mass m. The natural frequency of the undamped oscillator is $\omega_o \equiv \sqrt{k/m}$. For weak damping, A_ω peaks near $\omega = \omega_o$, with a peak width proportional to γ. And $\phi(\omega)$ is approximately 0^- for ω well below ω_o; approximately $-\pi$ well above ω_o; and equal to $-\pi/2$ at ω_o. $\phi(\omega)$ makes its transition from 0^- to $-\pi$ over a region of ω equal to the width of the peak in A_ω. $x_\omega(t)$ is known as the "persistent" or "steady-state" solution. It is a single sinusoidal function for t running from $-\infty$ to ∞.

In the absence of any external force, the weakly damped oscillator has a two-parameter family of free-decay solutions:

$$x_{\text{f-d}}(t) \approx B\, \mathrm{e}^{-\gamma t/2m} \cos(\omega_o t + \theta)$$

Those two parameters, B and θ, can be chosen, for example, to fit any particular initial conditions, i.e. any specified values for $x_{\text{f-d}}(0)$ and $\dot{x}_{\text{f-d}}(0)$.

In the presence of an externally applied force, linearity allows us simply to add a free-decay solution to the steady-state solution and choose B and θ to fit the forced solution to any desired values of the total $x(0)$ and $\dot{x}(0)$.

The next step is simply to realize that, if the externally applied force is zero for $t < 0$ and sinusoidal for $t \geq 0$, the steady-state plus transient solution constructed above is, indeed, the solution for $t \geq 0$, including the two "free" parameters that account for the particular initial conditions. The presence of the transient with frequency ω_o reflects that the sinusoidal force with frequency ω began abruptly at $t = 0$.

As stated, this is an elementary textbook problem. The solution presented here uses results and methods included in first and second year college physics courses. More advanced treatments include it as an example of (one-sided) Laplace transforms, i.e., to accommodate the $t \geq 0$ aspect.[3]

VI. ONE-ON-ONE TO MANY-ON-MANY

The solution for a single sinusoidal drive applied to a single damped oscillator can be used to construct the linear response of a damped, linear system to any driving force. The result is a double sum: Each Fourier component of the drive is applied to each normal mode of the damped system. Sophisticates will recognize this as the expansion of the relevant Green's function in terms of outer products of the eigenfunctions.

VII. QUALITATIVE FEATURES APPROPRIATE TO A PLUCK AT $t = 0$

In applying this to the pluck sound on a banjo (or any acoustic string instrument), the damped oscillator is one of the modes of the bridge-head-pot system. The applied force is one of the Fourier components of the string motion as it forces the bridge. The conceptual division of the musical instrument into two distinct systems is a consequence of the orders of magnitude of the physical parameters involved. Indeed, viewed as a whole, the entire instrument has a single set of normal modes, and these are set in motion by the pluck. However, an obvious decomposition into the two parts is clear from the produced sound, i.e., as viewed in Fig. 1. There are the equally spaced tall lines, and there's everything else.

The damped oscillator is originally at rest, i.e., $\dot{x}(0) = 0$. The initial displacement $x(0)$ is determined by the static force (tension $\times$ slope) of a Fourier component of the string configuration at $t = 0$ balanced against the spring constant of the body mode. The clonk contribution arises from the mismatch of the steady-state solution to the actual initial condition.

The goal here is a qualitative understanding of the origin and structure of the body sounds produced by a pluck. While the ideal string gives an excellent first approximation to the modes of a real musical string, the body modes are invariably more complicated. Even the first plausible approximation to a banjo is rather different from an ideal circular membrane or drum head.[2] So the following extreme approximation does no real harm while providing an enormous simplification and an important insight.

To get a picture of the strength of the excitation of body modes that make up the clonk, we can ignore their damping — except precisely on resonance. The $\gamma \neq 0$ amplitudes at that particular point determine how the singularities of the undamped solutions are regularized. The simplifying result in the equations is that the driving phase shift is exactly 0 or $-\pi$, except precisely on resonance where it is $-\pi/2$. This gives a good approximation to the clonk modes' initial amplitudes. And we just keep in mind that the clonk dies off faster than the string harmonics. Again, the basic approximations being made here are suggested by the actual sound of the instrument itself. They are not oversimplifications made just for convenience. The motion of the banjo head at the harmonic frequencies of the string is very long-lived and quite distinct from the head motion at intervening frequencies.

With regard to emphasizing $\omega_o \neq \omega$, those cases account for the clonk sound. Also, acoustic string instruments generally avoid having any body mode frequency precisely coincide with a played pitch or its harmonics. If those frequencies are too close, the played note will stand out as particularly loud and short lived.

Consider a string mode of frequency ω. We want to know the response of a body mode whose natural frequency is ω_0. There are two parts: steady state with frequency ω and amplitude A_ω and transient with amplitude B_ω. So it is natural to think of ω as some given value and solve for $A_\omega(\omega_0)$ and $B_\omega(\omega_0)$. I.e., how does a string mode excite a body mode? A further simplifying assumption is to consider a single m to describe different body modes as we consider how the excitation amplitudes depend on ω_0. (Suppressing any possible dependence on m is just for clarity of presentation and not an approximate feature of the inertia of body modes over the entire range of the instrument.)

Because we are taking $\gamma = 0$ for $\omega_o \neq \omega$, the phases are 0 or π. So we can put the phase shift into the sign of A_ω (and likewise for B_ω) to get a form appropriate for both ω_o less than and greater than ω. The familiar resonance response is:

$$A_\omega(\omega_o) = \frac{\mathtt{F}_\omega}{m} \frac{1}{\omega_o^2 - \omega^2}$$

for $\omega_o \neq \omega$, where $F_\omega(t) = \mathtt{F}_\omega \cos \omega t$. For $\omega_o = \omega$, there is a phase of $-\pi/2$ and an amplitude

$$A_\omega(\omega) = \frac{\mathtt{F}_\omega}{\omega\gamma} \ .$$

The sign of $F_\omega(t)$ is chosen so that the applied force is at its positive maximum at $t = 0$. (A plot of the full solution with $\gamma \neq 0$ only deviates noticeably from the $\gamma = 0$ curve over a tiny frequency region much, much narrower than the standard FWHF ["full width at half maximum"]; the deviation in that tiny region simply has the $\gamma = 0$ curve bend a bit to match the $A_\omega(\omega)$ value.)

To find $B_\omega(\omega_o)$, note that the steady-state solution and the total solution both have zero time derivative at $t = 0$. Hence, so does the transient. So there are no phases to worry about. The phases simply become + and – signs in the appropriate places.

For all ω_o, the total amplitude at $t = 0$ is determined by the displacement produced by a static ($\omega = 0$) external force:

$$x_{\text{total}}(0) = A_0(\omega_o) = \tfrac{\mathtt{F}_\omega}{m}\tfrac{1}{\omega_o^2} = A_0(\omega_o) + B_0(\omega_o) \ .$$

Hence, $B_0(\omega_o) = 0$.

For $\omega \neq 0$, $B_\omega(\omega_o)$ varies to cancel the variation of $A_\omega(\omega_o)$ in $x_{\text{total}}(0)$. In particular, for $\omega_o \neq \omega$,

$$B_\omega(\omega_o) = \tfrac{\mathtt{F}_\omega}{m}\tfrac{\omega^2}{\omega_o^2}\tfrac{1}{\omega_o^2-\omega^2} \ .$$

At resonance, i.e., $\omega_o = \omega$,

$$B_\omega(\omega) = F_\omega \ \sqrt{(\tfrac{1}{\omega\gamma})^2 + (\tfrac{1}{m\omega_o^2})^2}$$

with a phase of the obvious arctan.

VIII. TAKE AWAYS

Because of the resonance factor $\frac{1}{\omega_o^2-\omega^2}$, a string mode with frequency ω effectively excites body modes whose frequencies ω_o are comparable. These excited transients with frequencies ω_o have amplitudes that are initially comparable to the "steady" frequency ω motion. In addition, the $\frac{\omega^2}{\omega_o^2}$ factor means that significant transient excitation continues for all $\omega_o < \omega$. In the double sum required to describe a real instrument, the $\frac{\omega^2}{\omega_o^2}$ factor contributes to the clonk intensity piling up at lower frequencies seen in Fig. 1.

(A separate but relevant issue is the power that the string puts into individual ω's. An ideal pluck of an ideal string has a power spectrum that falls like $1/\omega^2$. However, the sharpness of the pluck geometry [e.g., as set by picks, nails, or finger flesh] sets the scale above which the actual power falls off much faster. So, there is negligible power above frequencies that correspond to a string wavelength on the scale of the sharpness.)

IX. WAVEFORMS

For orientation, I plot what these sorts of functions look like. To highlight the waveforms, i.e., shapes rather than absolute values, I show four graphs with the same ω and different values of ω_o. The horizontal time scale is the same in each graph and shows 50 cycles of the steady ω drive. The 1/e decay time of the ω_o clonk is taken to be 12 cycles of the ω oscillation — the same for each graph. (That's an unrealistically small value, chosen for illustrative purpose.)

The vertical scales are different for the different graphs. That's a conscious choice to emphasize shape over absolute values. I let the plotting software "autoscale" the graphs to fit the maximum value in the same size window. Each of the functions plotted has an overall, time-independent factor of $1/(\omega_o^2 - \omega^2)$, and that is **not** faithfully represented when comparing the figures.

Fig. 2 shows the sum of transient and steady state displacement where ω_o is 20% bigger than the driving ω. Fig. 3 is the case where it is 20% less.

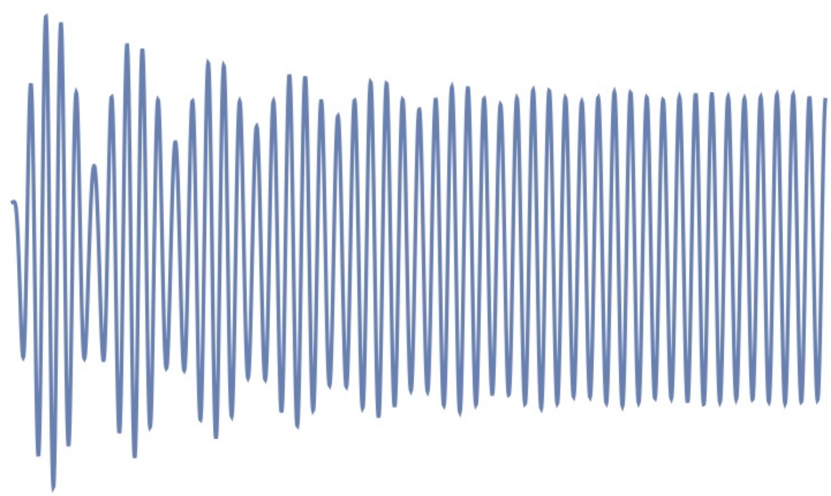

FIG. 2. total response with $\omega_o = 1.2 \times \omega$

It's evident that the initial transient and steady-state amplitudes are comparable, producing beats in the total, and then the transient dies away. Note that the $t = 0$ initial condition specified that x_{total} is positive and its derivative is zero. The net force is discontinuous at $t = 0$, jumping from zero for $t < 0$ to something positive for $t = 0^+$. In this approach, that is clear from the solution.

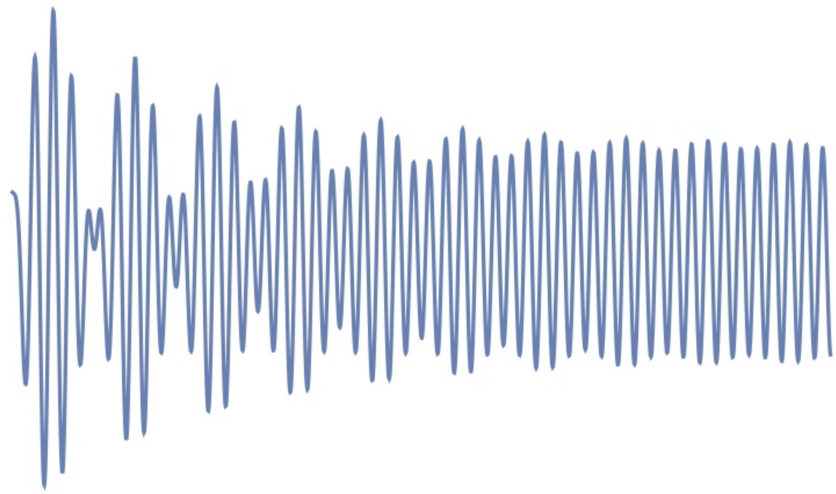

FIG. 3. total response with $\omega_o = 0.8 \times \omega$

I occasionally get confused in thinking what aspect of the pluck is represented by the oscillator in this discussion. The correct answer is a mode of the bridge & soundboard system. The string itself is at its maximum amplitude at $t = 0$, and it is applying its maximum upward force on the bridge. However, the bridge itself may be at a very small amplitude relative to its subsequent motion in the production of sound. In fact, the solution with $x_{\text{total}}(0) = 0$ as an initial condition is barely distinguishable from the ones in Fig.s 2 and 3, except for the first quarter cycle. In the $x_{\text{total}}(0) = 0$ case, obviously the initial total force is positive. The initial motions in FIG.s 2 and 3 are in the + direction for the same reason.

Fig. 4 illustrates a case where $\omega_0 = 0.1 \times \omega$. Even when the steady state response is small because $\omega_o \ll \omega$, there is a substantial transient response from the ω_o body mode.

Fig. 5 shows $\omega_0 = 3 \times \omega$. Recall the comment about autoscaling the vertical axis. The lesson is that relative contribution of the the clonk is negligible for $\omega_0 = 3 \times \omega$.

If the motion of Fig.s 2, 3, and 4 were turned directly into sound (with parameter values appropriate to a stringed instrument), we would hear the two frequencies as two pitches, and one dies away (much quicker than the other). That is also what would appear in a spectrogram. This is why plucked banjo strings produce sounds that are clearly not harmonious!

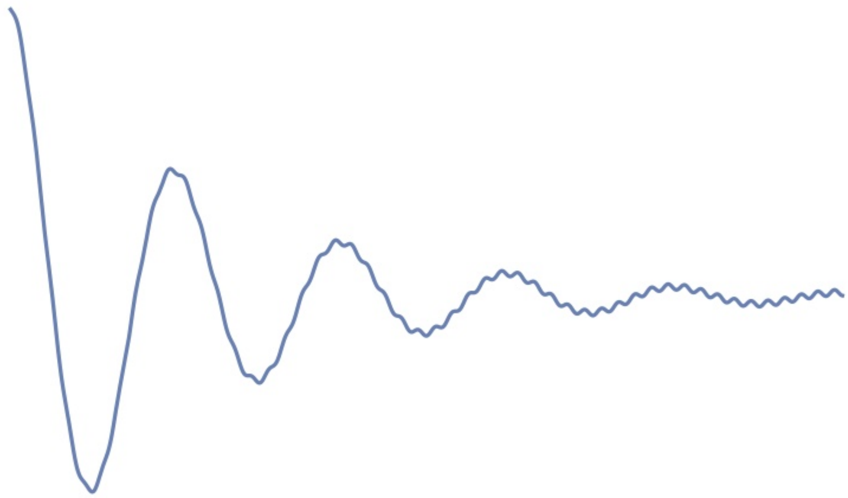

FIG. 4. total response with $\omega_o = 0.1 \times \omega$

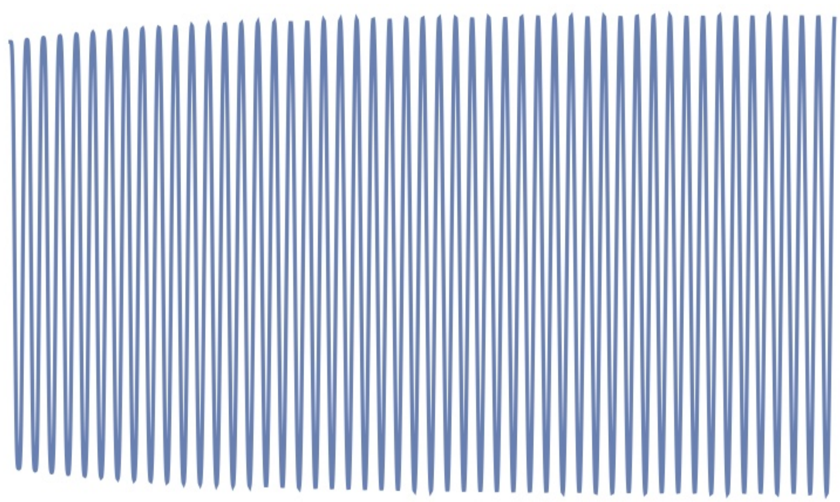

FIG. 5. total response with $\omega_o = 3 \times \omega$

X. SUMMARY

A plucked string imparts forces on the bridge at the frequency of its pitch and its harmonics. The relative strengths of the various harmonics are determined by the shape of the pluck — foremost, where it is plucked. In addition, the sudden onset of those forces excites the bridge much like a sudden tap, producing the initial clonk contribution to the sound. A tap excites the bridge/soundboard/body modes with roughly equal strengths over a broad range of frequencies. A string pluck is a little more particular as to where in frequency its clonk energy goes. It mostly feeds clonk frequencies near its own frequency and all frequencies below. But the effect is qualitatively similar to a tap in that a great many transients are excited. On wood-topped instruments, the clonk dies off very quickly relative to the sustain and the spacing in time of the played notes. It contributes an important part of the "attack" sound. However, on drum-headed instruments, the clonk lasts a substantial fraction of that time. Hence, it has a substantial role in perceived timbre.

XI. AFTERTHOUGHT

The matters discussed above are only a piece of the story of how a plucked instrument gets its voice. The other essential pieces include how much of the strings' initial energy is lost to heat and how efficient is the coupling to sound. These impact the strength and sustain of the various components of the sound. On wood-topped instruments, heat is the main end product of the energy budget, and the story is complicated. On drum-topped instruments, almost all the energy ultimately goes into sound, and most of that process can be successfully modeled with straightforward linear physics.[2]

[1] *On recognising violins: starting transients and the precedence effect*, J. Woodhouse, Catgut Acoustical Society Newsletter **39** 22-24 (1983); retrievable from https://stacks.stanford.edu/file/druid:gc245jb9875/CAS_gc245jb9875.pdf .

[2] *Acoustics of the Banjo: measurements and sound synthesis & theoretical and numerical modelling*, J. Woodhouse, D. Politzer, and H. Mansour, Acta Acustica, **5**, 15 and 16 (2021) https://doi.org/10.1051/aacus/2021009 and https://doi.org/10.1051/aacus/2021008; included are links to synthesized sound samples based on measurements and modeling, which are posted at https://euphonics.org; specifically, the synthesized banjo sounds are all in §5.5; more generally, euphonics.org is an extensive resource on the science of musical instruments, currently being assembled by author J.W. A much briefer and less technical version is *Pickers' Guide to* Acoustics of the Banjo:..., by D. Politzer, J. Woodhouse, and H. Mansour, HDP: 21 – 01, http://www.its.caltech.edu/~politzer, April 2021).

[3] e.g., E. Skudrzyk, *The Foundations of Acoustics*, Springer-Verlag (1971), §6.3 .